Definition 1 (The formal theory of β-equality)

Formulae The formulae of $\lambda\beta$ are just equations $M = N$ for all λ-term M and N.

Axioms (1) $\lambda x.M = \lambda y.[y/x]M$, whenever $y \notin FV(M)$;

(2) $(\lambda x.M)N = [N/x]M$;

(3) $M = M$

Inference Rules

$$\frac{M = M'}{NM = NM'}\ (\mu) \qquad \frac{N = N'}{NM = N'M}\ (\nu) \qquad \frac{M = M'}{\lambda x.M = \lambda x.M'}\ (\xi)$$

$$\frac{M = N}{N = M}\ (\sigma) \qquad \frac{M = N \quad N = L}{M = L}\ (\tau)$$

Definition 2 (The formal theory of weak equality)

Formulae The formulae of CLw are just equations $X = Y$ for all CL-term X and Y.

Axioms (1) $KXY = X$;

(2) $SXYZ = XZ(YZ)$;

(3) $X = X$

Inference Rules

$$\frac{X = X'}{ZX = ZX'}\ (\mu') \qquad \frac{Z = Z'}{ZX = Z'X}\ (\nu')$$

$$\frac{X = Y \quad Y = Z}{X = Z}\ (\tau') \qquad \frac{X = Y}{Y = X}\ (\sigma')$$

Definition 3 (The formal theory of β-reduction)

Formulae The formulae of $\lambda\beta$ are just equations $M \rhd N$ for all λ-term M and N.

Axioms (1) $\lambda x.M \rhd \lambda y.[y/x]M$, whenever $y \notin FV(M)$;

(2) $(\lambda x.M)N \rhd [N/x]M$;

(3) $M \rhd M$

Inference Rules

$$\frac{M \rhd M'}{NM \rhd NM'}\ (\mu) \qquad \frac{N \rhd N'}{NM \rhd N'M}\ (\nu) \qquad \frac{M \rhd M'}{\lambda x.M \rhd \lambda x.M'}\ (\xi)$$

$$\frac{M \rhd N \quad N \rhd L}{M \rhd L}\ (\tau)$$

Definition 4 (The formal theory of weak reduction)

Formulae The formulae of CLw are just equations $X \rhd Y$ for all CL-term X and Y.

Axioms (1) $KXY \rhd X$;

(2) $SXYZ \rhd XZ(YZ)$;

(3) $X \rhd X$

Inference Rules

$$\frac{X \rhd X'}{ZX \rhd ZX'}\ (\mu') \qquad \frac{Z \rhd Z'}{ZX \rhd Z'X}\ (\nu')$$

$$\frac{X \rhd Y \quad Y \rhd Z}{X \rhd Z}\ (\tau')$$

Lemma 5 *The following holds:*

(1) $M =_\beta N \quad \Leftrightarrow \quad \lambda\beta \vdash M = N$;

(2) $M \rhd_\beta N \quad \Leftrightarrow \quad \lambda\beta \vdash M \rhd N$;

(3) $M =_w N \quad \Leftrightarrow \quad CLw \vdash M = N$;

(4) $M \rhd_w N \quad \Leftrightarrow \quad CLw \vdash M \rhd N$.

Proof. By straightforward induction. I will show only (1), but the other parts are similarly proven.

($\Rightarrow$) By induction on β-reduction. Suppose that $M =_\beta N$ and $N \rhd^1_\beta L$. By I.H., we can assume that $\lambda\beta \vdash M = N$. If $N \rhd_\beta L$ is obtained from α-conversion, then $\lambda\beta \vdash M = L$ by rule (τ) and axiom (1). We will show the β-reduction case by subinduction on N.

- If $N \equiv z$ for some variable z, there is no β-redex.
- Suppose $N \equiv \lambda x P$. Then the main redex occurs in P, and there exists Q such that

  $$P \rhd_\beta Q$$
  $$L \equiv_\alpha \lambda x.Q.$$

  Thus we have $\lambda\beta \vdash M = L$ by rule (ξ), (τ) and I.H.
- Suppose $N \equiv PQ$. If the main redex is in P, then $\lambda\beta \vdash M = L$ by rule (ν), (τ) and I.H. If, on the other hans, the main redex is in Q. Similarly we have $\lambda\beta \vdash M = L$ by rule (μ), (τ) and I.H. Otherwise, PQ itself is the main redex. Then by Axiom (2) and (τ), we have $\lambda\beta \vdash M = L$.

($\Leftarrow$) Almost obvious by what we have learned from the class. ∎

Definition 6 Let σ be a first order vocabulary consisting of:

- Constant symbols S and K;
- 2-ary function symbol $*$.

Function symbol $*$ represent a concatenation of terms; e.g., $(S{*}K){*}(x{*}y)$ represents a CL-term, $SK(xy)$. In the following, we omit the functioni symbol $*$.

CL$^+$ consists of the following three axioms:

(1) $\forall x, y(Kxy = x)$;

(2) $\forall x, y, z(Sxyz = xz(yz))$;

(3) $S \neq K$.

We write CLw$^+$ $\vdash_{pc} X = Y$, if a formula $X = Y$ is derivable from CLw$^+$ by predicate calculus.

Lemma 7 (Barendregt) CLw$^+$ *is a conservative extention of CLw: i.e., for any CL-term* X *and* Y*, if* CLw$^+$ $\vdash_{pc} X = Y$ *then* CLw $\vdash X = Y$.

Proof. First of all, it is easily seen that CLw$^+$ is an extention of CLw. For instance, if CLw$^+$ $\vdash_{pc} X = X'$, then CLw$^+$ $\vdash_{pc} X * Y = X' * Y$, just because $*$ is a function symbol; therefore (ν') is valid.

For conservativity, suppose CLw$^+$ $\vdash_{pc} X = Y$. Then by soundness, $\mathfrak{A} \models X = Y$ for any model $\mathfrak{A}$ of CLw$^+$.

Consider the following first order structure $\mathfrak{A}$:

> the domain A of $\mathfrak{A}$ is the set of all equivalence class $X/\approx$ of CL-terms, where the equivalence relation $\approx$ is defined by:
>
> $$X \approx Y \Leftrightarrow \text{CLw} \vdash X = Y.$$

Then it sufficies to show that $\mathfrak{A}$ is a model of CLw$^+$; for, if $\mathfrak{A}$ is a model then, by soundness, CLw$^+$ $\vdash X = Y$ implies $\mathfrak{A} \models X = Y$ and thus CLw $\vdash X = Y$. However, it is obvious that $\mathfrak{A}$ is a model of CLw$^+$. The proof is completed. ∎

Definition 8 Suppose we are given the set F of all formulae in any means: this means just we are give a certain set. Then a inference rule over F is a partial function $R\colon F^{\alpha} \simeq F$, where α is an ordinal (or cardinal); but we will consider only the case where $\alpha \leq \omega$. We call a set $\mathcal{I}$ of inference rules a formal theory.

For instance, suppose $R\colon F^n \simeq F$, $\langle A_1, \ldots, A_n \rangle \in dom(R)$, and $R(A_1, \ldots, A_n) = A$. Then we say that we have a inference rule:

$$\frac{A_1, \ldots, A_n}{A}\ (R)_{.}$$

We call inferencerules R with no premises (i.e., $dom(R) = F^0 = \{\emptyset\}$) axioms.

Let $\mathcal{I}$ be a set of inference rules over F. We can define as usual the derivation of A from $\langle A_\beta \}_{\beta<\alpha}$ in $\mathcal{I}$; we write this as $\langle A_\beta \rangle_{\beta<\alpha} \vdash_{\mathcal{I}} A$.

Definition 9 Let $\mathcal{I}$ be a set of inference rules, and let $R\colon F^{\alpha} \simeq F$ be a inference rule. Then we say R is derivable in $\mathcal{I}$, if for any $\langle A_\beta \rangle_{\beta<\alpha} \in dom(R)$ we have $\langle A_\beta \rangle_{\beta<\alpha} \vdash_{\mathcal{I}} R(\langle A_\beta \rangle_{\beta<\alpha})$.

Definition 10 Let $\mathcal{I}$ be a set of inference rules, and let $R\colon F^{\alpha} \simeq F$ be a inference rule. Then we say R is admissible in $\mathcal{I}$, if the following holds:

> for any $\langle A_\beta \rangle_{\beta<\alpha} \in dom(R)$, if we have $\vdash_{\mathcal{I}} A_\beta$ for all $\beta < \alpha$, then $\vdash_{\mathcal{I}} R(\langle A_\beta \rangle_{\beta<\alpha})$.

Lemma 11

(1) R is admissible in $\mathcal{I}$, iff the theorems of $\mathcal{I} \cup \{R\}$ coincides with the theorems of $\mathcal{I}$: i.e.,

$$Th(\mathcal{I}) := \{\phi \in F \mid \vdash_{\mathcal{I}} \phi\} = Th(\mathcal{I} \cup \{R\}) := \{\phi \in F \mid \vdash_{\mathcal{I}} \cup \{R\} \phi\}.$$

(2) If R is derivable in $\mathcal{I}$, then R is admissible in $\mathcal{I}$, but not vice versa.

(3) If R is derivable in $\mathcal{I}$, then R is also derivable in any extention of $\mathcal{I}$.

Proof. (1) ($\Rightarrow$) Suppose R is admissible in $\mathcal{I}$ and A is a theorem of $\mathcal{I} \cup \{R\}$. We will show that A is also a theorem of $\mathcal{I}$ by induction on the deduction of A. If the last inference is R, then since the premises are all theorem of $\mathcal{I}$ by I.H. and R is admissible, A is a theorem of $\mathcal{I}$. Otherwise, the last inference is in $\mathcal{I}$ and obviously A is a theorem of $\mathcal{I}$ by I.H.

($\Leftarrow$) Let $\langle A_\beta \mid \beta < \alpha \rangle \in dom(R)$. If A_β is a theorem of $\mathcal{I}$ for all $\beta < \alpha$, then they are also a theorem of $\mathcal{I} \cup \{R\}$ and thus $A = R(A_\beta)_{\beta<\alpha}$ is a theorem of $\mathcal{I} \cup \{R\}$. Then by the assumption, A is a theorem of $\mathcal{I}$; this means that R is admissible in $\mathcal{I}$.

(2) Obvious.

(3) Obvious.

∎

Lemma 12 *In the previous lemma, the opposit direction in (2) does not hold.*

Proof. Trivial. Let $\mathcal{I} = \emptyset$. Then every inference rule is trivially admissible. But every inference rule trivially non-derivable. ∎

Example. Let add just four new constants a, b, c and d to $\lambda\beta$; that is,

$$Term(\lambda\beta) ::= a|b|c|d|x|MN|\lambda zM.$$

Let $\lambda\beta^*$ denote this new system.

Consider the following new rule R.

$$\frac{a = b}{c = d}\, R$$

In fact, R is admissible in $\lambda\beta'$, since $a = b$ is not a theorem in $\lambda\beta$. However, R is not derivable in $\lambda\beta'$, since we cannot derive $c = d$, even if we assume $a = b$.

This fact is proven by showing (modified) Curch-Rosser Theorem holds for $\lambda\beta$.

Definition 13 Let $\mathcal{I}$ and $\mathcal{I}'$ be formal theories (i.e., sets of inference rules) with the same set of formulae. We say $\mathcal{I}$ and $\mathcal{I}'$ are therem-equivalent, iff every inference rules in $\mathcal{I}$ is admissible in $\mathcal{I}'$, and vice versa. We say $\mathcal{I}$ and $\mathcal{I}'$ are rule-equivalent, iff every inference rules in $\mathcal{I}$ is derivable in $\mathcal{I}'$, and vice versa.

Lemma 14 *Let $\mathcal{I}$ and $\mathcal{I}'$ be formal theories with the same set of formulae. Then, $\mathcal{I}$ and $\mathcal{I}'$ are theorem-equivalent, iff they have the same set of theorems: i.e., $\vdash_{\mathcal{I}} A \Leftrightarrow \vdash_{\mathcal{I}'} A$.*

Proof. By easy and straightforward induction on deduction. ∎

Definition 15 Let $\mathcal{I}$ be a formal theory, and let some of its formulae be of the form $X = Y$. Then the equality relation determined by $\mathcal{I}$, written by $=_{\mathcal{I}}$, is defined by

$$X =_{\mathcal{I}} Y \quad :\Leftrightarrow \quad \vdash_{\mathcal{I}} X = Y.$$

Lemma 16 *Let $\mathcal{I}$ and $\mathcal{I}'$ be formal theories with the same formulae. Suppose it includes some formulae of the form $X = Y$. Then, if $\mathcal{I}$ and $\mathcal{I}'$ are theorem-equivalent, then they give the same equality relation.*

Proof. Immediate from the previour lemma. ∎