The Undecidability Theorem

Our main goal is to prove that β-equality for λ-terms, as well as weak equality for CL-terms, are recursively undecidable; that is, there is no recursive algorithm that can decide whether or not two terms are equivalent.

We will use the following notation for Church numerals:

$$\overline{n} = \lambda fx.f^n x.$$

Since our arguments will work equally for λ-terms and CL-terms, we will use $=_{\beta,w}$ to denote either β- or weak equality.

We will also assume that a Gödel numbering on terms is given, denoted $gd(\cdot)$. The numbering should be such that there exist recursive total functions τ, ν such that

$$\tau(gd(X), gd(Y)) = gd(XY)$$

and

$$\nu(n) = gd(\bar{n}).$$

n Let $\ulcorner X \urcorner = \overline{gd(X)}$.

Definition (recursively separable):

Two sets A and B of natural numbers are *recursively separable* iff there is a recursive total function ϕ whose only values are 0 and 1, such that

$$n \in A \Rightarrow \phi(n) = 1,$$

$$n \in B \Rightarrow \phi(n) = 0.$$

Definition (closed under equality):

A set A of terms is *closed under equality* iff, for all terms X and Y,

$$X =_{\beta,w} Y \Rightarrow (X \in A \Rightarrow X \in B).$$

Scott-Curry undecidability theorem

Theorem 1. *For λ-terms and β-equality, or CL-terms and weak equality, no pair of non-empty sets of terms which are closed under equality is recursively separable.*

Suppose ϕ separates A and B, where A and B are disjoint sets of terms that are non-empty and closed under equality. Let F define ϕ, so that

$$X \in A \Rightarrow F\ulcorner X \urcorner =_{\beta,w} \overline{1},$$

$$X \in B \Rightarrow F\ulcorner X \urcorner =_{\beta,w} \overline{0}.$$

Let T define τ and N define ν, so that

$$T\ulcorner X\urcorner\ulcorner Y\urcorner =_{\beta,w} \ulcorner XY\urcorner$$

and

$$N\overline{n} =_{\beta,w} \ulcorner\overline{n}\urcorner.$$

Now choose terms X in A and Y in B. We will construct a term J which depends on X and Y such that

$$F^{\ulcorner} J^{\urcorner} =_{\beta,w} \overline{1} \Rightarrow J =_{\beta,w} B,$$

$$F^{\ulcorner} J^{\urcorner} =_{\beta,w} \overline{0} \Rightarrow J =_{\beta,w} A.$$

This will cause a contradiction because, letting $j = gd(J)$, we shall have

$$
\begin{aligned}
\phi(j) = 1 \ &\Rightarrow \ F\ulcorner J \urcorner =_{\beta,w} \overline{1} \\
&\Rightarrow \ J =_{\beta,w} Y \\
&\Rightarrow \ J \in B \\
&\Rightarrow \ \phi(j) = \overline{0}
\end{aligned}
$$

and

$$
\begin{aligned}
\phi(j) = 0 \ &\Rightarrow \ F\ulcorner J \urcorner =_{\beta,w} \overline{0} \\
&\Rightarrow \ J =_{\beta,w} X \\
&\Rightarrow \ J \in A \\
&\Rightarrow \ \phi(j) = \overline{1}.
\end{aligned}
$$

Let D be the pairing term we constructed such that

$$DXY\overline{1} =_{\beta,w} Y,$$

$$DXY\overline{0} =_{\beta,w} X.$$

We want to build J such that

$$J =_{\beta,w} DXY(F\ulcorner J\urcorner).$$

Define

$$J \equiv H\ulcorner H \urcorner,$$

where

$$H \equiv \lambda y.DXY(F(Ty(Ny))).$$

We then have

$$\begin{array}{rcl} J & =_{\beta,w} & DXY(F(T\ulcorner H\urcorner(N\ulcorner H\urcorner))) \\ & =_{\beta,w} & DXY(F(T\ulcorner H\urcorner\ulcorner\ulcorner H\urcorner\urcorner)) \\ & =_{\beta,w} & DXY(F\ulcorner(H\ulcorner H\urcorner)\urcorner) \\ & \equiv & DXY(F\ulcorner J\urcorner). \end{array}$$

Corollary 1. *If a set A of terms is closed under equality, it is not recursive.*

Corollary 2. *The set of all terms which have normal form is not recursive.*

Corollary 3. *There is no recursive total ψ such that*

$$\psi(gd(X), gd(Y)) = \begin{cases} 1 & \text{if } X =_{\beta,w} Y, \\ 0 & \text{if } X \neq_{\beta,w} Y. \end{cases}$$